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Journal of Approximation Theory 122 (2003) 13–23

JOURNAL OF Approximation Theory

http://www.elsevier.com/locate/jat

Lebesgue constant for the Strömberg wavelet

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Received 25 March 2002; accepted 27 January 2003

Communicated by Carl de Boor

Abstract

Let S denote the Strömberg wavelet in $L^2(\mathbb{R})$ and $P_{s,n}$ $(s\in\mathbb{Z},\ n\in\mathbb{Z}\cup\{\infty\})$, the orthogonal projection onto the space spanned by the functions $2^{r/2}S(2^r t - m)$, where $r\leqslant s$, $m<n+1$ (i.e. $P_{s,n}$ are partial sums for the orthonormal wavelet basis generated by S). We show that the maximum of the norms of the extensions of the operators $P_{s,n}$ onto $L^\infty(\mathbb{R})$ is equal to $2+(2-\sqrt{3})^2$.

Keywords: Strömberg wavelet; Lebesgue constant; Best constant; Piecewise linear functions; Orthogonal projections

1. Introduction

The goal of this paper is to compute the exact value of the Lebesgue constant for the Strömberg wavelet system of functions in the space $L^\infty(\mathbb{R})$. It is known since [3] that the norm of orthogonal projection onto a space of piecewise linear functions is always bounded in L^∞ by 3 and in the general case this constant is optimal (see [6]). Here, it is shown that in the particular case of partial sum subspaces for the orthonormal Strömberg wavelet basis the lowest upper bound is equal to $2+(2-\sqrt{3})^2$. The calculation of this result reduces essentially to finding the L^∞-norm of orthogonal projection onto the subspace of continuous piecewise linear functions with respect to the dyadic partition of the real line (i.e. for $i\in\mathbb{Z}$, one has knots in i if $i\geqslant 0$ and in $i/2$ if $i<0$). It should also be noted that in [4], Ciesielski formulated a

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doi:10.1016/S0021-9045(03)00030-3

hypothesis that the Lebsgue constant for the Franklin system in the space $C[0,1]$ has the same value.

This paper is arranged in the following manner: In this section, the necessary facts concerning spaces of piecewise linear functions on the real line and the Strömberg wavelet are provided. In Section 2, the theorem from which the value of the constant follows is formulated and proved.

1.1. Notation and general facts

This section introduces the basic notation used in the paper and presents some facts concerning piecewise linear functions on the real line. The theorems presented here are either elementary or well known to specialists; brief proofs are provided to make the paper self-contained.

Throughout the paper, $L^p(\mathbb{R})$ denotes the space L^p on $\mathbb{R}$ with one-dimensional Lebesgue measure and $\langle \cdot, \cdot \rangle$ stands for the scalar product in the Hilbert space $L^2(\mathbb{R})$.

Let $\pi = (t_i)_{i \in \mathbb{Z}}$ denote a partition of the real line $\mathbb{R}$, i.e. $(t_i)_{i \in \mathbb{Z}}$ is an increasing sequence of real numbers such that $\lim_{i \to \pm\infty} t_i = \pm\infty$. Define $\delta_i = t_i - t_{i-1}$ and let $\mathscr{S}_\pi$ denote the subspace of $L^2(\mathbb{R})$ consisting of all continuous piecewise linear functions with knots in the points t_i. Finally, let P_π stand for the orthogonal projection in $L^2(\mathbb{R})$ onto $\mathscr{S}_\pi$.

For a given partition π, define $\Lambda_i \in \mathscr{S}_\pi$, $i \in \mathbb{Z}$, as functions satisfying the conditions $\Lambda_i(t_j) = \delta_{i,j}$ $(i, j \in \mathbb{Z})$. Every function $f \in \mathscr{S}_\pi$ can be expressed in the form

$$f(t) = \sum_{i \in \mathbb{Z}} f(t_i) \Lambda_i(t). \tag{1.1}$$

Because $\operatorname{supp} \Lambda_i = [t_{i-1}, t_{i+1}]$, the sum is finite for every $t \in \mathbb{R}$. Let $(\Lambda_i^*)_{i \in \mathbb{Z}}$ denote the system of functions in $\mathscr{S}_\pi$ biorthogonal to $(\Lambda_i)_i$ (i.e. $\langle \Lambda_i^*, \Lambda_j \rangle = \delta_{i,j}$). Since the restriction of $\mathscr{S}_\pi$ to any finite interval is finite-dimensional, $\mathscr{S}_\pi$ is a closed subspace of $L^2(\mathbb{R})$ and any linear functional on $\mathscr{S}_\pi$ whose support is in a finite interval is continuous. In particular, $\xi_i(f) := f(t_i)$ is a continuous linear functional on $\mathscr{S}_\pi$, and the existence of the system $(\Lambda_i^*)_{i \in \mathbb{Z}}$ follows from the Riesz representation theorem. (But see also Remark 1.2.) Expansion (1.1) of $P_\pi f$ has the form

$$P_\pi f = \sum_{i \in \mathbb{Z}} \langle f, \Lambda_i^* \rangle \Lambda_i(t), \tag{1.2}$$

from which it follows that

$$||P_\pi||_\infty = \sup_{i \in \mathbb{Z}} \int_{\mathbb{R}} |\Lambda_i^*(t)| \, dt. \tag{1.3}$$

It was first shown by Ciesielski in [3] that for any partition of the interval $[0,1]$, the norm of the corresponding orthogonal projection treated as an operator in $L^\infty([0,1])$ does not exceed 3. The same holds in the case of infinite partitions of the real line. Moreover, let $a_{i,j} := \Lambda_i^*(t_j)$. Then the bi-infinite matrix $(a_{i,j})$ is checkerboard. These two facts are expressed in the following proposition:

Proposition 1.1. *Let* $(a_{i,j})$ *and* P_π *be defined as above. Then*

$$||P_\pi||_\infty \leqslant 3, \tag{1.4a}$$

$$a_{i,j} = (-1)^{i+j}|a_{i,j}| \quad \text{for } i, j \in \mathbb{Z} \tag{1.4b}$$

and

$$\frac{\delta_j}{6} a_{i,j-1} + \frac{\delta_j + \delta_{j+1}}{3} a_{i,j} + \frac{\delta_{j+1}}{6} a_{i,j+1} = \delta_{i,j} \quad \text{for } i, j \in \mathbb{Z}. \tag{1.4c}$$

Moreover, if $\inf_{i\in\mathbb{Z}} \delta_i > 0$, *then*

$$\lim_{j\to\pm\infty} a_{i,j} = 0, \quad i \in \mathbb{Z}. \tag{1.4d}$$

Remark 1.1. In fact, one can prove (1.4d) without the assumption $\inf \delta_i > 0$. For example, using a method similar to the one found in [5, Sections 6.3–6.4] it can be shown that the numbers $|a_{i,j}|$ decay exponentially as $j \to \pm\infty$ and $|a_{i,i}| = \max_j |a_{i,j}|$. However, as in this paper only partitions with interval lengths separated from zero are used, there is no need to complicate the proof by considering a more general case.

Remark 1.2. To prove the existence of the functions $(\Lambda_i^*)_{i\in\mathbb{Z}}$ one can also rely solely on the facts expressed in Proposition 1.1.

Proof. Let $\bar{\Lambda}_i \coloneqq 2\Lambda_i/(\delta_i + \delta_{i+1})$, so that $||\bar{\Lambda}_i||_1 = 1$. We consider the bi-infinite tri-diagonal system

$$\sum_{j\in\mathbb{Z}} \langle \bar{\Lambda}_i, \Lambda_j \rangle \xi_j(f) = \langle \bar{\Lambda}_i, f \rangle, \quad i \in \mathbb{Z},$$

with the right side well defined for any $f \in L^\infty(\mathbb{R})$. The matrix A of this system has the general row

$$\frac{1}{3}\frac{\delta_j}{\delta_j + \delta_{j+1}}, \; \frac{2}{3}, \; \frac{1}{3}\frac{\delta_{j+1}}{\delta_j + \delta_{j+1}}.$$

Thus, $||A||_\infty = 1$ and A is diagonally dominant, which implies it is bounded from below in the supremum norm by

$$\inf_{i\in\mathbb{Z}} \left(A(i,i) - \sum_{j\neq i} |A(i,j)| \right) = \frac{1}{3}.$$

This means that A is boundedly invertible on ℓ^∞ and $||A^{-1}||_1 \leqslant 3$. As $\Lambda_i^* = \sum_{j\in\mathbb{Z}} A^{-1}(i,j)\bar{\Lambda}_j$, it now follows that

$$||\Lambda_i^*||_1 \leqslant \sum_{j\in\mathbb{Z}} |A^{-1}(i,j)| \leqslant 3. \tag{1.5}$$

This implies (1.4a). By an argument similar to the one in [2, p. 459], A is totally positive and hence due to [1, Theorem 4.5], A^{-1} is checkerboard and (1.4b) follows.

Equalities (1.4c) follow from the equations $\langle \Lambda_i^*, \Lambda_j \rangle = \delta_{i,j}$ and (1.4d) follows from (1.5). $\square$

Now (1.2), (1.3) and (1.4a) imply

Corollary 1.2. *Every operator P_π can be extended onto the whole space $L^\infty(\mathbb{R})$, where for any $f \in L^\infty(\mathbb{R})$ the function $P_\pi(f)$ is given by* (1.2) *and the norm $\|P_\pi\|_\infty$ is preserved.*

From (1.4b) after a straightforward computation using the fact that $\Lambda_i^* \in S_\pi$ we obtain

Corollary 1.3.

$$\int_{\mathbb{R}} |\Lambda_i^*(t)|\, dt = \frac{1}{2} \sum_{j\in\mathbb{Z}} \delta_{j+1} \frac{|a_{i,j}|^2 + |a_{i,j+1}|^2}{|a_{i,j}| + |a_{i,j+1}|}. \tag{1.6}$$

The proof of the following is elementary:

Proposition 1.4. *Suppose that the partition $\tilde{\pi} = (\tilde{t}_i)_{i\in\mathbb{Z}}$ is a non-singular affine transformation of the partition π, i.e. $\tilde{\pi} = \lambda\pi + m$, where $\lambda \in \mathbb{R}\backslash\{0\}$, $m \in \mathbb{R}$, and $\tilde{t}_0 = \lambda t_0 + m$. If $\lambda > 0$ then*

$$\tilde{a}_{i,j} = \frac{1}{\lambda} a_{i,j} \quad \text{and} \quad \int_{\mathbb{R}} |\tilde{\Lambda}_i^*(t)|\, dt = \int_{\mathbb{R}} |\Lambda_i^*(t)|\, dt, \qquad i,j \in \mathbb{Z} \tag{1.7a}$$

while for $\lambda < 0$

$$\tilde{a}_{i,j} = \frac{1}{\lambda} a_{-i,-j} \quad \text{and} \quad \int_{\mathbb{R}} |\tilde{\Lambda}_i^*(t)|\, dt = \int_{\mathbb{R}} |\Lambda_{-i}^*(t)|\, dt, \qquad i,j \in \mathbb{Z}. \tag{1.7b}$$

Hence, the norm $\|P_\pi\|_\infty$ is not affected by non-singular affine transformations of the partition π.

1.2. The Strömberg wavelet

The following partitions of the real line are significant from the point of view of this article:

$$\pi(0,0) := (t_i)_{i\in\mathbb{Z}}, \quad \text{where } t_i = \begin{cases} \dfrac{i}{2} & \text{for } i < 0, \\ i & \text{for } i \geqslant 0, \end{cases}$$

$$\pi(r,m) := 2^{-r}\pi(0,0) + m, \quad \text{for } r, m \in \mathbb{Z}$$

$$\pi(r,\infty) := 2^{-r-1}\mathbb{Z}, \quad \text{for } r \in \mathbb{Z}.$$

These partitions are ordered by inclusion. Specifically, for $r<s$ or $r=s$ and $m<n$ one has $\pi(r,m)\subset\pi(s,n)$ and $\mathscr{S}_{\pi(r,m)}\subset\mathscr{S}_{\pi(s,n)}$. For such pairs (r,m) and (s,n) the notation $(r,m)<(s,n)$ will be used. The operators $P_{\pi(r,m)}$ will be written as $P_{r,m}$.

The Strömberg wavelet (first introduced in [7]) is a function $S\in\mathscr{S}_{\pi(0,1)}$ such that $||S||_2=1$ and S is orthogonal to $\mathscr{S}_{\pi(0,0)}$. It is a known fact that if $S_{r,m}(t)=2^{r/2}S(2^r t-m)$, $r,m\in\mathbb{Z}$, then $(S_{r,m})_{r,m\in\mathbb{Z}}$ is an orthonormal basis in $L^2(\mathbb{R})$. Moreover, $(S_{r,m})_{r\leqslant s,m<n+1}$ is an orthonormal basis in the space $\mathscr{S}_{\pi(s,n)}$ for $s\in\mathbb{Z}$ and $n\in\mathbb{Z}\cup\{\infty\}$. With respect to the ordering of the subspaces $\mathscr{S}_{\pi(r,m)}$, the partial sum operators for the system $(S_{r,m})_{r,m\in\mathbb{Z}}$ are

$$P_{s,n}f=\sum\langle f,S_{r,m}\rangle S_{r,m}, \tag{1.8}$$

where the sum extends over all pairs $(r,m)\leqslant(s,n)$ if $n<\infty$ and $(r,m)<(s,\infty)$ if $n=\infty$. By Corollary 1.2, the operators $P_{r,m}$ as well as the formulas (1.8) extend onto $L^\infty(\mathbb{R})$.

2. Lebesgue constant for the Strömberg wavelet

In this section, we prove that the Lebesgue constant for the extensions of operators $P_{r,m}$ onto $L^\infty(\mathbb{R})$ is equal to $2+(2-\sqrt{3})^2$. This is stated in the theorem below.

Theorem 2.1. *The partial sum operators $P_{r,m}$ for the system $(S_{r,m})_{r,m\in\mathbb{Z}}$ satisfy*

$$||P_{r,m}||_\infty=2+(2-\sqrt{3})^2 \quad \textit{if } r,m\in\mathbb{Z}, \tag{2.9a}$$

$$||P_{r,\infty}||_\infty=2 \quad \textit{if } r\in\mathbb{Z}. \tag{2.9b}$$

Hence,

$$\sup_{(r,m)}||P_{r,m}||_\infty=2+(2-\sqrt{3})^2. \tag{2.9c}$$

Proof. Because the norms $||P_{r,m}||_\infty$ are not affected by affine transformations of the partitions $\pi(r,m)$, it suffices to show that

$$||P_{-1,\infty}||_\infty=2 \quad \text{and} \quad ||P_{0,1}||_\infty=2+(2-\sqrt{3})^2.$$

These norms can be calculated using formula (1.3) and Corollary 1.3 in the case of partitions $\pi=\pi(-1,\infty)$ and $\pi=\pi(0,1)$.

In order to simplify the formulas that appear in the remaining part of the paper, we will always assume that $\alpha:=-2+\sqrt{3}$ and $\beta:=2-\sqrt{3}=|\alpha|$.

Lemma 2.2. *Let $\lambda > 0$ and π be a partition of the real line such that $t_0 = 0$, $\delta_i = 1$ for $i > 0$ and $\delta_i = \lambda$ for $i \leqslant 0$. Then for $i \geqslant 0$*

$$a_{i,j} = \begin{cases} \dfrac{2\sqrt{3}}{1+\lambda}\alpha^{i-j} & \text{for } j \leqslant 0, \\ \dfrac{(1-\lambda)\sqrt{3}}{1+\lambda}\alpha^{i+j} + \sqrt{3}\alpha^{i-j} & \text{for } 0 \leqslant j \leqslant i, \\ \dfrac{(1-\lambda)\sqrt{3}}{1+\lambda}\alpha^{i+j} + \sqrt{3}\alpha^{-i+j} & \text{for } j \geqslant i \end{cases} \tag{2.10a}$$

and for $i \leqslant 0$

$$a_{i,j} = \begin{cases} \dfrac{-(1-\lambda)\sqrt{3}}{\lambda(1+\lambda)}\alpha^{-i-j} + \dfrac{\sqrt{3}}{\lambda}\alpha^{i-j} & \text{for } j \leqslant i, \\ \dfrac{-(1-\lambda)\sqrt{3}}{\lambda(1+\lambda)}\alpha^{-i-j} + \dfrac{\sqrt{3}}{\lambda}\alpha^{-i+j} & \text{for } 0 \geqslant j \geqslant i, \\ \dfrac{2\sqrt{3}}{1+\lambda}\alpha^{-i+j} & \text{for } j \geqslant 0. \end{cases} \tag{2.10b}$$

Proof. The method used to find the coefficients $a_{i,j}$ has very much in common with the calculation in [7, Section 4] of the values of the Strömberg wavelet in the knots of the dyadic partition of the real line.

We first consider the case $i \geqslant 0$. Eqs. (1.4c) for $j < 0$, $0 < j < i$ and $j > i$ give recursive formulas, respectively, for the sequences $(a_{i,j})_{j=0}^{\infty}$, $(a_{i,j})_{j=0}^{i}$, $(a_{i,j})_{j=i}^{\infty}$ of the form $a_{i,j-1} + 4a_{i,j} + a_{i,j+1} = 0$, where j falls into one of the intervals indicated above. Each of these systems has a two-parameter family of solutions $(u\alpha^j + v\alpha^{-j})_j$, where α and α^{-1} are the roots of the polynomial $x^2 + 4x + 1$. Hence, there exist numbers $u_1, v_1, u_2, v_2, u_3, v_3$ such that

$$a_{i,j} = \begin{cases} u_1\alpha^j + v_1\alpha^{-j} & \text{for } j \leqslant 0, \\ u_2\alpha^j + v_2\alpha^{-j} & \text{for } 0 \leqslant j \leqslant i, \\ u_3\alpha^j + v_3\alpha^{-j} & \text{for } j \geqslant i, \end{cases} \tag{2.11}$$

Formulas in (2.11) must coincide for $j = 0$ and $j = i$. Moreover Eqs. (1.4c) for $j = 0$ and $j = i$ must be satisfied. Finally (1.4d) must hold, which implies $u_1 = v_3 = 0$. Together, there are six linear equations for the coefficients $u_1, v_1, u_2, v_2, u_3, v_3$. Solving them leads to the formulas in (2.10a). (2.10b) may be obtained from (2.10a) using Proposition 1.4. $\square$

From the lemma we can now calculate the norm $||P_{-1,\infty}||_{\infty}$. Namely, we set $\lambda = 1$. In this case, $a_{0,j} = \sqrt{3}\alpha^{|j|}$. Due to Proposition 1.4,

$$\begin{aligned}\int_{\mathbb{R}} |N_i^*(t)|\,dt &= \int_{\mathbb{R}} |N_0^*(t)|\,dt \\ &= \frac{1}{2}\sum_{j=-\infty}^{\infty} \frac{|a_{0,j}|^2 + |a_{0,j+1}|^2}{|a_{0,j}| + |a_{0,j+1}|} = \sum_{j=0}^{\infty} \frac{|a_{0,j}|^2 + |a_{0,j+1}|^2}{|a_{0,j}| + |a_{0,j+1}|} \\ &= \sqrt{3}\,\frac{1+\beta^2}{1+\beta}\sum_{j=0}^{\infty}\beta^j = \sqrt{3}\,\frac{1+\beta^2}{1-\beta^2} \\ &= 2,\end{aligned}$$

because $(1+\beta^2)/(1-\beta^2) = 2/\sqrt{3}$.

The calculation of $||P_{0,1}||_{\infty}$ is more complicated. We now set $\lambda = 1/2$ and $\pi = \pi(0,1)$.

Lemma 2.3. *Let* $A_i = \int_{\mathbb{R}} |\Lambda_i^*(t)|\,dt$. *Then for* $i \geqslant 0$

$$A_i = 1 + \frac{2}{3}\beta^i + \frac{1}{3}\beta^{2i} + \frac{\sqrt{3}}{2}\beta^i \sum_{j=0}^{i-1} p_j, \tag{2.12a}$$

$$A_{-i} = 1 + \frac{4}{3}\beta^i - \frac{1}{3}\beta^{2i} + \frac{\sqrt{3}}{2}\beta^i \sum_{j=0}^{i-1} q_j, \tag{2.12b}$$

where

$$\begin{aligned} p_j &= \frac{(1+\beta^{-2})\beta^{-2j} + 4/3 + 1/9(1+\beta^2)\beta^{2j}}{(1+\beta^{-1})\beta^{-j} + 1/3(1+\beta)\beta^j}, \\ q_j &= \frac{(1+\beta^{-2})\beta^{-2j} - 4/3 + 1/9(1+\beta^2)\beta^{2j}}{(1+\beta^{-1})\beta^{-j} - 1/3(1+\beta)\beta^j}. \end{aligned} \tag{2.13}$$

Proof. We first substitute $\lambda = 1/2$ in (2.10a) and obtain for $i \geqslant 0$

$$|a_{i,j}| = \begin{cases} \left(\dfrac{4\sqrt{3}}{3}\beta^i\right)\beta^{-j} & \text{for } j \geqslant 0, \\ \beta^i\left(\dfrac{\sqrt{3}}{3}\beta^j + \sqrt{3}\beta^{-j}\right) & \text{for } 0 \leqslant j \leqslant i, \\ \left(\dfrac{\sqrt{3}}{3}\beta^i + \sqrt{3}\beta^{-i}\right)\beta^j & \text{for } j \geqslant i. \end{cases} \tag{2.14a}$$

We have used the fact that $|(\sqrt{3}/3)\alpha^k + \sqrt{3}\alpha^{-k}| = (\sqrt{3}/3)\beta^k + \sqrt{3}\beta^{-k}$ for any $k \in \mathbb{Z}$.

Similarly, for $i \leqslant 0$,

$$|a_{i,j}| = \begin{cases} 2\left(-\dfrac{\sqrt{3}}{3}\beta^{-i} + \sqrt{3}\beta^{i}\right)\beta^{j} & \text{for } j \leqslant i, \\ 2\beta^{-i}\left(-\dfrac{\sqrt{3}}{3}\beta^{-j} + \sqrt{3}\beta^{j}\right) & \text{for } 0 \geqslant j \geqslant i, \\ \left(\dfrac{4\sqrt{3}}{3}\beta^{-i}\right)\beta^{j} & \text{for } j \geqslant 0. \end{cases} \tag{2.14b}$$

This time we have used the equality $|-(\sqrt{3}/3)\alpha^{-k} + \sqrt{3}\alpha^{k}| = -(\sqrt{3}/3)\beta^{-k} + \sqrt{3}\beta^{k}$, which holds for any non-positive integer k.

Now, we compute A_i for $i \geqslant 0$ using Corollary 1.6 and (2.14a):

$$\begin{aligned} A_i =& \frac{1}{2}\sum_{j\in\mathbb{Z}} \delta_{j+1}\frac{|a_{i,j}|^2 + |a_{i,j+1}|^2}{|a_{i,j-1}| + |a_{i,j}|} \\ =& \underbrace{\frac{1}{2}\sum_{j<0}\frac{1}{2}\frac{|a_{i,j}|^2 + |a_{i,j+1}|^2}{|a_{i,j}| + |a_{i,j+1}|}}_{X_i} + \underbrace{\frac{1}{2}\sum_{0\leqslant j<i}\frac{|a_{i,j}|^2 + |a_{i,j+1}|^2}{|a_{i,j}| + |a_{i,j+1}|}}_{Y_i} \\ &+ \underbrace{\frac{1}{2}\sum_{j\geqslant i}\frac{|a_{i,j}|^2 + |a_{i,j+1}|^2}{|a_{i,j}| + |a_{i,j+1}|}}_{Z_i}. \end{aligned}$$

We calculate

$$\begin{aligned} X_i =& \frac{1}{2}\sum_{j<0}\frac{1}{2}\frac{\beta^{-2j}(4(\sqrt{3}/3)\beta^{i})^2 + \beta^{-2j-2}(4(\sqrt{3}/3)\beta^{i})^2}{\beta^{-j}(4(\sqrt{3}/3)\beta^{i}) + \beta^{-j-1}(4(\sqrt{3}/3)\beta^{i})} \\ =& \frac{1}{4}\frac{4\sqrt{3}}{3}\beta^{i}\sum_{j\geqslant 0}\frac{\beta^{2j} + \beta^{2j+2}}{\beta^{j} + \beta^{j+1}} = \frac{\sqrt{3}}{3}\beta^{i}\frac{1+\beta^2}{1+\beta}\sum_{j\geqslant 0}\beta^{j} \\ =& \frac{\sqrt{3}}{3}\frac{1+\beta^2}{1-\beta^2}\beta^{i} = \frac{2}{3}\beta^{i}. \end{aligned}$$

The last equality is obtained by evaluating $(1+\beta^2)/(1-\beta^2)$ with $\beta = 2-\sqrt{3}$:

$$\begin{aligned} Y_i =& \frac{1}{2}\sum_{j=0}^{i-1}\frac{\beta^{2i}((\sqrt{3}/3)\beta^{j} + \sqrt{3}\beta^{-j})^2 + \beta^{2i}((\sqrt{3}/3)\beta^{j+1} + \sqrt{3}\beta^{-j-1})^2}{\beta^{i}((\sqrt{3}/3)\beta^{j} + \sqrt{3}\beta^{-j}) + \beta^{i}((\sqrt{3}/3)\beta^{j+1} + \sqrt{3}\beta^{-j-1})} \\ =& \frac{\sqrt{3}}{2}\beta^{i}\sum_{j=0}^{i-1}\frac{(1+\beta^{-2})\beta^{-2j} + 4/3 + 1/9(1+\beta^2)\beta^{2j}}{(1+\beta^{-1})\beta^{-j} + 1/3(1+\beta)\beta^{j}} \\ =& \frac{\sqrt{3}}{2}\beta^{i}\sum_{j=0}^{i-1}p_j. \end{aligned}$$

$$
\begin{aligned}
Z_i &= \frac{1}{2}\sum_{j\geqslant i}\frac{((\sqrt{3}/3)\beta^i+\sqrt{3}\beta^{-i})^2\beta^{2j}+((\sqrt{3}/3)\beta^i+\sqrt{3}\beta^{-i})^2\beta^{2j+2}}{((\sqrt{3}/3)\beta^i+\sqrt{3}\beta^{-i})\beta^j+((\sqrt{3}/3)\beta^i+\sqrt{3}\beta^{-i})\beta^{j+1}}\\
&= \frac{1}{2}\left(\frac{\sqrt{3}}{3}\beta^i+\sqrt{3}\beta^{-i}\right)\sum_{j\geqslant i}\frac{\beta^{2j}+\beta^{2j+2}}{\beta^j+\beta^{j+1}}\\
&= \frac{\sqrt{3}}{2}\left(\frac{1}{3}\beta^i+\beta^{-i}\right)\frac{1+\beta^2}{1+\beta}\sum_{j\geqslant 0}\beta^j = 1+\frac{1}{3}\beta^{2i}.
\end{aligned}
$$

To obtain the last equality we have again replaced all powers of β which do not depend on i with the same power of $2-\sqrt{3}$. Summing up we get, for $i\geqslant 0$,

$$
A_i = X_i + Y_i + Z_i = 1 + \frac{2}{3}\beta^i + \frac{1}{3}\beta^{2i} + \frac{\sqrt{3}}{2}\beta^i\sum_{j=0}^{i-1} p_j.
$$

The formula for A_{-i} is obtained in a similar manner using (2.14b) or by application of Proposition 1.4. $\square$

It is important to note that both the numerators and the denominators in the formulas for p_j and q_j are positive for $j\geqslant 0$. Moreover, the following holds:

Lemma 2.4. *Let p_j and q_j for $j\geqslant 0$ be defined as in* (2.13). *Then $p_j \leqslant q_j$.*

Proof. For fixed $j\geqslant 0$ set

$$
u := (1+\beta^{-2})\beta^{-2j},\quad v := 4/3,\quad w := 1/9(1+\beta^2)\beta^{2j},
$$
$$
x := (1+\beta^{-1})\beta^{-j},\quad y := 1/3(1+\beta)\beta^j.
$$

The numbers u, v, w, x, y are all positive. We have to show the inequality

$$
\frac{u+v+w}{x+y}\leqslant\frac{u-v+w}{x-y},
$$

which, as $x-y>0$ and $u-v+w>0$, is equivalent to

$$
xv\leqslant y(u+w).
$$

The verification of the last inequality is elementary. $\square$

Note that from Lemma 2.3 we immediately get $A_0=2$. We now prove that $A_{-i}\geqslant A_{-i-1}$ for $i\geqslant 1$. By (2.12b), this inequality is equivalent to

$$
(\beta^{-1}-1)\left(\frac{4}{3}-\frac{1}{3}(1+\beta)\beta^i+\frac{\sqrt{3}}{2}\sum_{j=0}^{i-1} q_j\right)\geqslant\frac{\sqrt{3}}{2}q_i. \tag{2.15}
$$

Neglecting the negative summand in the denominator of q_j, we see that

$$
q_j\geqslant\frac{1}{\beta^{-1}+1}\left((1+\beta^{-2})\beta^{-j}-\frac{4}{3}\beta^j+\frac{1}{9}(1+\beta^2)\beta^{3j}\right). \tag{2.16}
$$

We now estimate every q_j for $j = 0, \dots, i-1$ using (2.16), sum up the obtained finite geometric series and replace all fixed powers of β with the same power of $2-\sqrt{3}$, arriving finally at

$$\frac{\sqrt{3}}{2}\sum_{j=0}^{i-1} q_j \geqslant (\beta^{-i}-1) - \frac{1}{3}(1-\beta^i) + \frac{1}{45}(1-\beta^{3i}). \tag{2.17}$$

Therefore, to prove (2.15) it suffices to show that

$$(\beta^{-1}-1)\left(\beta^{-i} + \frac{1}{45} - \frac{1}{3}\beta^{i+1} - \frac{1}{45}\beta^{3i}\right) \geqslant \frac{\sqrt{3}}{2}\,\frac{(1+\beta^{-2})\beta^{-2i} - 4/3 + 1/9(1+\beta^2)\beta^{2i}}{(1+\beta^{-1})\beta^{-i} - 1/3(1+\beta)\beta^i}. \tag{2.18}$$

Note again that the numerator as well as the denominator on the right are positive. We multiply both sides by the denominator on the right, group together the terms with β^{-2i}, β^{-i-1}, β^i, β^{-1+2i} and β^{4i}, explicitly evaluate the coefficients obtained for each of the above-listed powers of β by substituting $\beta = 2-\sqrt{3}$ and finally multiply both sides of the inequality by $45\sqrt{3}/2$. After this rather straightforward but tedious calculation, we arrive at the following inequality, equivalent to (2.18):

$$3\beta^{-1-i} + \beta^{4i} \geqslant 45 + \beta^i + 3\beta^{-1+2i}. \tag{2.19}$$

For $i = 1$, (2.19) does not hold. However, using (2.12b) one may directly verify that

$$A_{-2} = \frac{2150 - 1156\sqrt{3}}{73} < A_{-1} = 9 - 4\sqrt{3} = 2 + (2-\sqrt{3})^2.$$

When $i \geqslant 2$,

$$3\beta^{-1-i} + \beta^{4i} > 3\beta^{-3} > 49 > 45 + \beta^2 + 3\beta^3 \geqslant 45 + \beta^i + 3\beta^{-1+2i}.$$

The monotonicity of $(A_{-i})_{i\geqslant 0}$ is therefore established. Recall that $A_0 = 2$ and observe that, for $i \geqslant 1$, $A_i \leqslant A_{-i}$, because

$$\tfrac{2}{3}\beta^i + \tfrac{1}{3}\beta^{2i} \leqslant \tfrac{4}{3}\beta^i - \tfrac{1}{3}\beta^{2i} \quad \text{and by Lemma 2.4 } p_j \leqslant q_j.$$

Hence, the inequality $A_i \leqslant A_{-i}$ follows from formulas (2.12) in Lemma 2.3. Finally,

$$\|P_{0,1}\|_\infty = \sup_{i\in\mathbb{Z}} A_i = A_{-1} = 2 + (2-\sqrt{3})^2. \qquad \square$$

Acknowledgments

The result presented here was a part of the author's M.Sc. thesis written under the supervision of Professor Przemysław Wojtaszczyk at the Faculty of Mathematics, Informatics and Mechanics of the Warsaw University. The author would also like to thank an anonymous referee for valuable advice on the proof of Proposition 1.1 and the suggestion to provide more details in the proof of Lemma 2.3, as well as some other remarks concerning the material presented in this paper.

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